Second Order Analysis of Hollow Tapered Circular Bridge Pier

S. A. Halkude and D.L.Mittapalli
Walchand Institute of Technology, Solapur – 413006, Maharastra, India
E-mail: halkude60@gmail.com, dmittapalli@gmail.com

(Received on 20 August 2012 and accepted on 05 November 2012)

***Abstract* - The Beams subjected to the axial force and lateral force simultaneously are known as beam-columns. Bridge pier is idealized as a column subjected to axial load and biaxial moment. Slender member subjected to axial force and biaxial bending moment fails due to buckling effect. This buckling is caused due to slenderness effect also known as 'PΔ' effect. The objective of the research reported in this paper is to obtain a theoretical formulation, using beam column theory for studying the behavior of straight hollow circular section and tapered hollow circular section of the bridge pier. Study is carried for different heights of bridge pier for straight hollow circular pier and tapered hollow circular pier. Study is carried by considering two lane box type bridge girder. Providing a straight hollow circular section for a direct & flexural action proves to be uneconomical. The straight hollow circular section of bridge pier is replaced by a tapered hollow circular pier section in the present study.**

***Keywords:* Beam-column theory, Second order analysis, PΔ effect, Bridge pier, Slenderness effect, Tapered hollow Circular bridge pier**

I. Introduction

Bridge Piers are subjected to forces in longitudinal direction as well as in transverse direction. This force causes biaxial moment at base of the pier. The pier is idealized as a column subjected to axial force and biaxial moment. These moments and axial force cause the pier to buckle along its longitudinal direction. This buckling is nothing but deflection of the pier. If the base moment due to these deflections is not considered then, it is known as first order analysis. By the first order analysis the structural capacity of the pier is estimated approximately. In order to get more accurate results, second order analysis of bridge pier is to be done, where the buckling effect is considered. Beam column theory is used for second order analysis.

Bridge pier is subjected to an axial load and biaxial moments. Iterative neutral axis method is used to analyze the pier. Section is subjected to axial force combined with two orthogonal moments. The working load analysis is to assume as 'mono axial bending with axial force' and on this mono axially cracked section the effect of other orthogonal moment is superimposed.

II. Second Order Analysis Using Beam-Column Theory

Beams subjected to the axial compression and simultaneously supporting lateral loads are known as beam-columns. The basic equation for the analysis of beam-column is derived by considering the beam in Fig 1. The beam is subjected to an axial compressive force P and to a distributed lateral load of intensity q which varies with the distance 'x' along the beam. Consider an element of length 'd_x' between the two cross sections taken normal to the original axis of the beam as shown in the Fig 2. The lateral load may be considered as having constant intensity q over the distance 'd_x' and will be assumed positive when in the direction of the positive y axis which is downward in this case. The shearing force V and bending moment M acting on the sides of the elements are assumed positive in the directions down. The relations among load, shearing force B, and bending moments are obtained from the equilibrium of the element in Fig 2 Summing forces in the y direction give:

$$-V + qdx + (V + dv) = 0$$

Or $$q = -\frac{dV}{dx} \qquad (2.1)$$

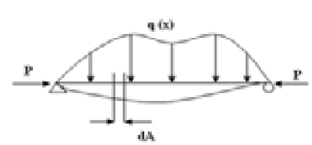

Fig 1. General loading beam-column analysis.

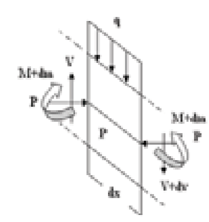

Fig 2. Cross section of beam.

By assuming that the angle between the axis of the beam and the horizontal is small we obtain,

$$M + qdx\frac{dx}{2} + (V + dv) - (M + dM) + P\frac{dy}{dx}dx = 0$$

If the terms of second-order are neglected, this equation becomes

$$V = \frac{dM}{dx} - P\frac{dy}{dx} \qquad (2.2)$$

*If the effects of shearing deformations and shortening of the beam axis are neglected the expression for the curvature of the a*xis of the beam is,

$$EI\frac{d^2y}{dx^2} = -M \qquad (2.3)$$

Where, EI represents the flexural rigidity of the beam in the plane of bending that is, in the XY plane, which is assumed to be plane of symmetry. Combining equation (2.3) with equation (2.1) and equation (2.2) we can express the differential equations of the axis of the beam in the following alternate forms,

$$EI\frac{dy}{dx} + P\frac{dy}{dx} = -V \qquad (2.4)$$

$$EI\frac{dy}{dx} + P\frac{dy}{dx} = q \qquad (2.5)$$

Equations (2.1) to (2.5) are the basic differential equations for bending considering beam-column. If the axial force's P equals zero, these equations reduces to the usual equations for bending by lateral loads only. The nature of the axial forces has significant effect on the deflections and ultimately on the secondary moments.

III. Theorotical Formulation

Analysis of pier fixed at base and hinged at top subjected to axial load and uniaxial bending by using beam column theory for different loading conditions:

3.1 Trapezoidal Load Throughout the Height of Pier

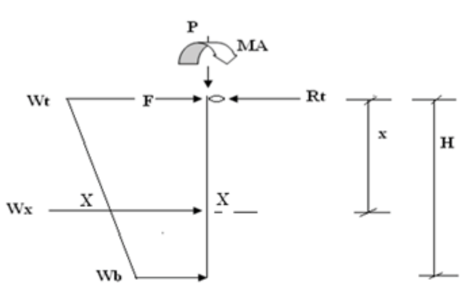

Fig 3. Loading on pier

Lateral load intensity at general section

'x' on pier $W_x = W_T - \left[\frac{(W_T - W_B)x}{H}\right]$ (3.1.1.1)

3.1.1. First Order Analysis of Pier

Let 'M_x' be the bending moment at a general section 'XX' at a distance 'x' from top of pier,

$$\therefore \quad M_x = -M - Fx + R_T - \frac{W_T x^2}{2} + \frac{(W_T - W_B)x^3}{6H} \qquad (3.1.1.2)$$

Using strain energy method to claculate the bending moment equation,

$$R_T = \frac{3M}{2H} + F + \frac{11W_T H}{40} + \frac{W_B H}{10}$$

$$M_x = -M + \frac{3M}{2H}x + \frac{11W_T H}{40}x + \frac{W_B H}{10} - \frac{W_T x^2}{2} + \frac{(W_T - W_B)x^3}{6H} \qquad (3.1.1.3)$$

3.1.2. Second Order Analysis of Pier

Considering the same values used in first order analysis as given above:

Bending moment at a general section 'x' is given by

$$\therefore M_x = Py - M_A + (R_T - F)x - \frac{W_T x^2}{2} + \frac{k_w x^3}{6} \qquad (3.1.2.1)$$

$$y = A\sin(\alpha x) + B\cos(\alpha x)$$

$$-\frac{k_w}{6P}x^3 + \frac{W_T}{2P}x^2 + \frac{x}{P}\left[F - R_T + \frac{k_w}{\alpha^2}\right] + \frac{1}{P}\left[M_A - \frac{W_T}{\alpha^2}\right] \qquad (3.1.2.2)$$

On substituting the values and using boundary conditions,

At x=0, y=0 in equation 3.1.2.2 we get,

$$\therefore B = -\frac{1}{P}\left[M_A - \frac{W_T}{\alpha^2}\right]$$

On substituting the values and using boundary conditions, at

$x = H, y = 0, x = H, \frac{\partial y}{\partial x} = 0$ in equation 3.1.2.2 we get,

$$\therefore A = \frac{1}{\sin(\alpha H)}\left[\frac{k_w}{6P}H^3 - B\cos(\alpha H) - \frac{W_T}{2P}H^2 - \frac{H}{P}\left[F - R_T + \frac{k_w}{\alpha^2}\right] - \frac{1}{P}\left[M_A - \frac{W_T}{\alpha^2}\right]\right]$$

(3.1.2.3)

$$R_T = \frac{1}{\frac{1}{P}\left[\frac{\tan(\alpha H)}{\alpha} - H\right]} \times$$

$$\left[\begin{array}{l}\frac{W_T}{P}H\left\{\frac{\tan(\alpha H)}{\alpha}-\frac{H}{2}\right\}-B\{\sin(\alpha H)\tan(\alpha H)+\cos(\alpha H)\}-\frac{k_w H}{P\alpha}\left\{\frac{H\tan(\alpha H)}{2}+\frac{1}{\alpha}\right\} \\ +\frac{\tan(\alpha H)}{\alpha P}\left\{F+\frac{k_w}{\alpha^2}\right\}+\frac{k_w}{6P}H^3+\frac{W_T}{P\alpha^2}-\frac{HF}{P}-\frac{M}{P} \qquad (3.1.4)\end{array}\right]$$

$$A = \frac{\pi}{4}(D^2-d^2)$$

$$I = \frac{\pi}{64}[(d_1)^2-(d_2)^2]$$

3.2 Trapezoidal Load at a General Height of the Pier

This analysis is done in two parts

i. For axial compressive load and lateral forces

ii. For axial compressive load and bending moment (applied moment) at top of pier.

3.2.1 For Axial Compressive Load and Lateral Forces

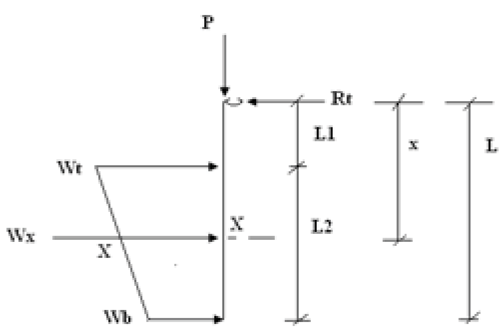

Fig 4. Axial compressive load and lateral forces

Lateral load intensity at general section 'x' on pier

$$W_x = W_B + \frac{(W_T - W_B)(L-x)}{(L-l)}$$

Second order analysis for axial load and trapezoidal lateral load by using Beam-Column Analysis:

$$M_x = Py - M_B + (W_x - R_{T1})x \qquad (3.2.1.1)$$

We have, $EI_x \frac{\partial^2 y}{\partial x^2} = -M_x$

$$EI_x \frac{\partial^2 x}{\partial y^2} = -R_{T1} - Py \qquad (3.2.1.2)$$

On solving the above equation (3.2.1.2) for constants we get

$$y = A\cos(\alpha x) + B\sin(\alpha x) - \frac{R_{T1}}{P}x \qquad (3.2.1.3)$$

$$y = C\cos(\alpha x) + D\sin(\alpha x) + \frac{M_B}{P} + \frac{x}{P}(R_{T1} - W_x) \qquad (3.2.1.4)$$

On substituting the values and using boundary conditions,

At x=0 , y=0

in equation (3.2.1.4), we get

$$\therefore D = -\frac{M_B}{P}$$

$$x = 0, \frac{\partial y}{\partial x} = 0$$

$$\because C = -\frac{1}{P\alpha}(R_{T1} - W_x)$$

$$y = \frac{1}{P}(R_{T1} - W_x)\left\{x - \frac{\sin(\alpha x)}{\alpha}\right\} + \frac{M_B}{P}\{1 - \cos(\alpha x \qquad (3.2.1.5)$$

On substituting the values and using boundary conditions,
At x=0 , y=0

In equation (3.2.1.3), we get

$$\therefore B = 0$$

$$\therefore y = A\sin(\alpha x) - \frac{R_{T1}}{P}x \qquad (3.2.1.6)$$

At $x = a$,

Deflection and slope of the column remains same on both side

Comparing equation (3.2.1.5) and (3.2.1.6)

$$\therefore A\sin(\alpha a) - \frac{R_{T1}}{P}a$$

$$= \frac{1}{P}(R_{T1} - W_x)\left\{X - \frac{\sin(\alpha X)}{\alpha}\right\} + \frac{M_B}{P}\{1 - \cos(\alpha X)\}$$

Where, $X = (H - a)$

On solving above equation we get

$$\therefore A\sin(\alpha a) - \frac{R_{T1}}{P}\left\{a + \left[X - \frac{\sin(\alpha X)}{\alpha}\right] - H[1 - \cos(\alpha X)]\right\}$$

$$= \frac{W_x}{P}\left\{X[1-\cos(\alpha X)] - \left[X - \frac{\sin(\alpha X)}{\alpha}\right]\right\} \quad (3.2.1.7)$$

Now comparing slopes of equations (3.2.1.5) and (3.2.1.6)

$$A\alpha\cos(\alpha a) - \frac{R_{T1}}{P} = C\alpha\cos(\alpha X) - D\alpha\sin(\alpha X) + \frac{1}{P}(R_{T1} - W_x)$$

On solving above equation we get

$$A\alpha\cos(\alpha a) \quad -\frac{R_{T1}}{P}\ [2-\cos(\alpha X) - H\alpha\sin(\alpha X)]$$

$$= \frac{W_x}{P} \quad \{\cos(\alpha X) + X\alpha\sin(\alpha X) - 1\} \quad (3.2.1.8)$$

Solving equation (3.2.1.7) and (3.2.1.8) simultaneously the value of ' R_{T1} '

$$R_{T1} = \frac{\left[W_x\left\{\left(\frac{\alpha}{\tan(\alpha a)}\left(X\left(1-\cos(\alpha X) - \left(X - \frac{\sin(\alpha X)}{\alpha}\right)\right)\right) - (\cos(\alpha X) + \alpha X\sin(\alpha X) - 1)\right)\right\}\right]}{\left[\{2-\cos(\alpha X) - H\alpha\sin(\alpha X)\} - \frac{\alpha}{\tan(\alpha a)}\left\{a + \left(X - \frac{\sin(\alpha X)}{\alpha}\right) - H(1-\cos(\alpha X))\right\}\right]}$$

(3.2.1.9)

$A = \frac{\pi}{4}(D^2 - d^2)$

$I = \frac{\pi}{64}[(d_1)^2 - (d_2)^2]$

Since, the resultant force ' W_x ' is varying along the column from l_1 to l_2. On integrating R_{T1} along the length we can get value of reactions and substituting this value we can calculate other constants of integration A, C, and D respectively. And substituting the all values the deflections at any general point can be calculated

3.3.1 For Axial Compressive Load and Bending Moment

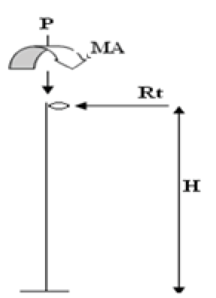

Fig 5 Axial loading and bending moment

$$M_B = M_A - R_{T2}H$$

At a general section 'XX' at a distance 'x' from top, bending moment is given by

$$M_x = Py - M_A + R_{T2}x$$

$$y = y_c + y_p$$

$$y_c = A\sin(\alpha x) + B\cos(\alpha X)$$

$$y_p = -\frac{R_{T2}}{P}x + \frac{M_A}{P}$$

$$\therefore y = A\sin(\alpha x) + B\cos(\alpha x) - \frac{R_{T2}}{P}x + \frac{M_A}{P} \quad (3.2.2.1)$$

On substituting the values and using boundary conditions,

At x=0 , y=0

In equation (3.2.2.1),

$$\therefore B = -\frac{M_A}{P}$$

$$x = H, y = 0$$

$$\therefore A\sin(\alpha H) - \frac{R_{T2}}{P}H = -\frac{M_A}{P}(1-\cos(\alpha H)) \quad (3.2.2.2)$$

$$x = H, \ \frac{\partial y}{\partial x} = 0$$

$$\therefore A\alpha\cos(\alpha H) - \frac{R_{T2}}{P} = -\frac{M_A}{P}\alpha\sin(\alpha H) \quad (3.2.2.3)$$

On solving above equations (3.2.2.2) and (3.2.2.3) simultaneously we get

$$R_{T2} = \frac{\left[\frac{M_A}{P}\alpha\left\{\sin(\alpha H) - \frac{1}{\tan(\alpha H)}(1-\cos(\alpha H))\right\}\right]}{\left[1 - \frac{\alpha H}{\tan(\alpha H)}\right]} \quad (3.2.2.4)$$

From above equations (3.2.1.9) and (3.2.2.4)

We get total reaction ' R_T ' for the column.

$A = \frac{\pi}{4}(D^2 - d^2)$

$I = \frac{\pi}{64}[(d_1)^2 - (d_2)^2]$

IV. Parametric Work

The forces on the bridge pier are calculated as specified in IRC and the maximum moment on a bridge pier for different heights is calculated. Using the combined stress equation and considering stresses constant the behavior for straight hollow circular pier & tapered hollow circular pier is calculated. The volume of concrete required for different height of bridge pier for straight hollow circular pier & tapered hollow circular pier is calculated. Accordingly, the percentage saving in concrete for different height of bridge pier for tapered hollow circular pier with respect to straight hollow circular pier is calculated. The Variation of Slenderness for different height of pier for straight hollow pier and tapered hollow circular pier is calculated. For tapered hollow circular pier variation of slope for different height of bridge pier is calculated. Cost comparison for different height of bridge pier for Tapered hollow circular pier and straight hollow circular pier is calculated.Cost of pier is calculated by considering material cost and form work cost of bridge pier.

Table I Volume of Concrete Required for Different Height of Bridge Pier For Straight Hollow Circular Pier & Tapered Hollow Circular Pier.

Height	Base Bending Moment	Straight Hollow circular pier		Tapered Hollow Circular Pier				Volume of Concrete Required for	
		External	Internal	Base section External	Base section Internal	Top section External	Top section Internal	Straight Hollow	Tapered Hollow
m	kN.m	D1	D2	D1	D2	d1	d2	m^3	m^3
15.00	6823	2760.00	1380.00	2760.00	1380.00	1931.00	1380.00	67.27	42.26
20.00	12238	3236.00	1618.00	3236.00	1618.00	2103.00	1618.00	123.30	70.25
25.00	19027	3675.00	1837.50	3675.00	1837.50	2274.00	1837.50	198.79	105.89
30.00	27537	4104.00	2052.00	4104.00	2052.00	2448.00	2052.00	297.49	150.40
35.00	37608	4514.00	2257.00	4514.00	2257.00	2621.00	2257.00	419.88	203.93
40.00	48918	4897.00	2448.50	4897.00	2448.50	2786.00	2448.50	564.74	265.74

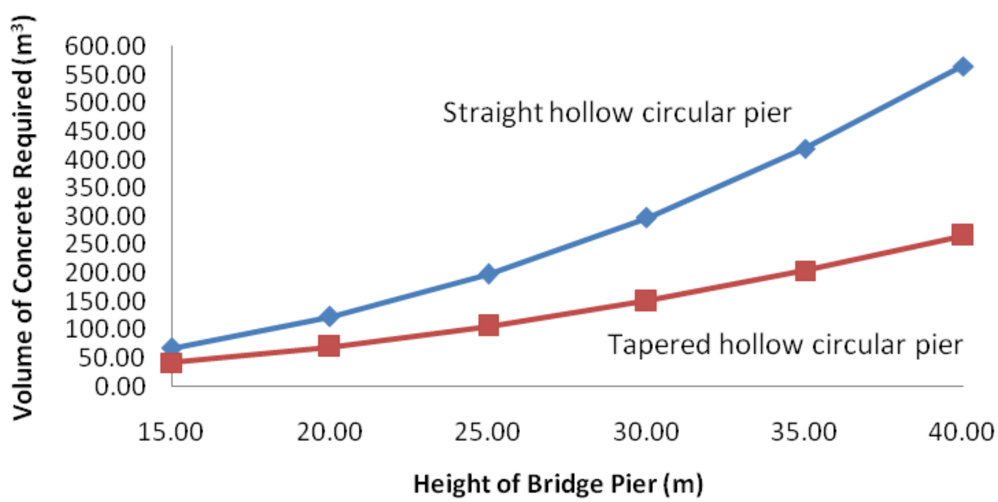

Fig.4.1 The variation in volume of concrete required for different height of bridge pier for straight hollow circular pier & tapered hollow circular pier

Volume of concrete required increases as the height of the bridge pier increases. The Rate of increase in volume of concrete required is milder for tapered hollow circular pier in comparison with straight hollow circular pier. The volume of concrete required for tapered hollow circular pier is varying largely in comparison with volume of concrete required for straight hollow circular pier. (Refer Table I and Graph 4.1)

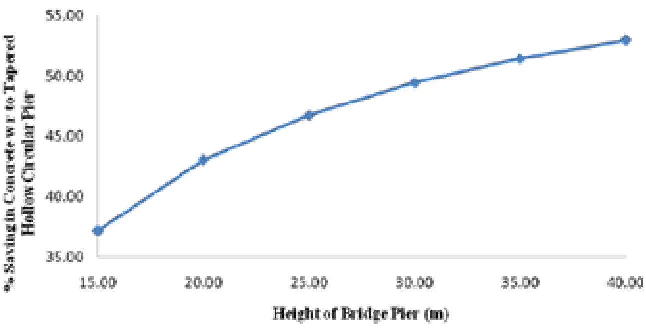

Fig. 4.2 The variation in percentage saving in concrete for different height of bridge pier for straight hollow circular pier & tapered hollow circular pier.

Percentage saving in concrete required for tapered hollow circular pier with respect to straight hollow circular pier increases as the height of bridge pier increases. Percentage saving of concrete required for Tapered hollow circular pier section with respect to straight hollow circular pier section increases and increase is nearly linear. (Refer Figure 4.2)

Volume of concrete required increases with increase in slenderness ratio for hollow circular pier and tapered hollow circular pier. Rate of increase in volume of concrete required is milder for tapered hollow circular pier in comparison with straight hollow circular pier. The rate of increase in slenderness ratio for Tapered hollow circular pier is milder in comparison with straight hollow circular pier.(Refer Table II and Figure 4.3)

As the designed base bending moment increases the slope of pier varies from steeper to milder. Slope of pier decreases as the height of bridge pier increases. Cross section of the piers required at the base increases as the height of bridge pier increases. (Refer Figure 4.4)

Table II The Variation of Slenderness for Different Height of Pier For Straight Hollow Pier and Tapered Hollow Circular Pier.

Height (m)	Base Bending Moment (kNm)	Volume of Concrete Required		Slenderness Ratio	
		Straight Hollow Circular Pier (m^3)	Tapered Hollow Circular Pier (m^3)	Straight Hollow Circular Pier	Tapered Hollow Circular Pier
15.00	6823	67.27	42.26	13.61	17.70
20.00	12238	123.30	70.25	15.48	21.10
25.00	19027	198.79	105.89	17.04	23.94
30.00	27537	297.49	150.40	18.31	26.30
35.00	37608	419.88	203.93	19.42	28.33
40.00	48918	564.74	265.74	20.46	30.20

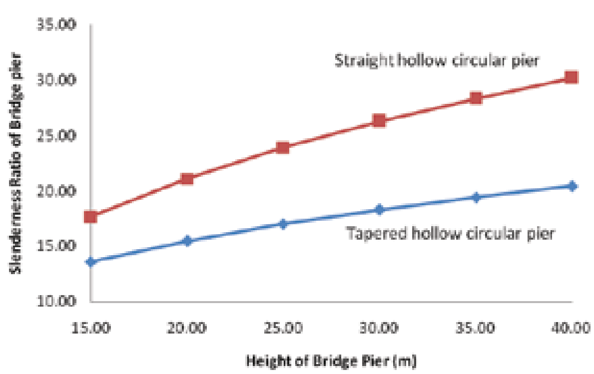

Fig. 4.3 The Variation of Slenderness ratio for different height of pier for straight hollow pier and tapered hollow circular pier

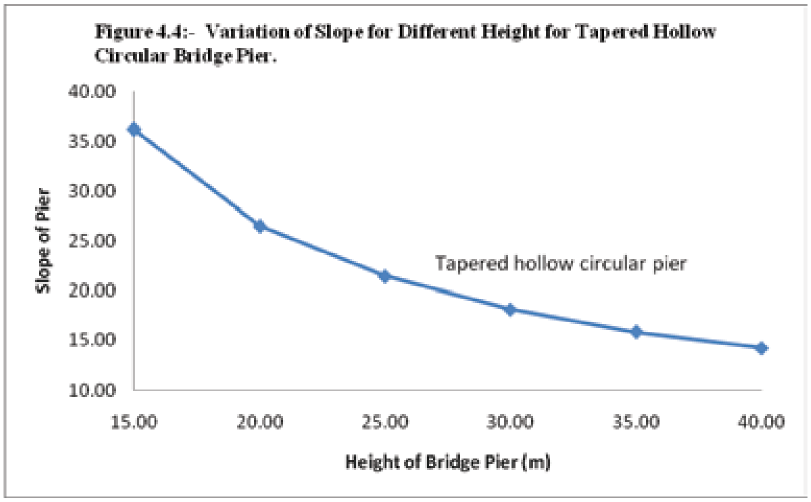

Fig.4.4 Variation of slope for different height of bridge pier for tapered hollow circular bridge pier

TABLE III COST COMPARISON FOR DIFFERENT HEIGHT OF BRIDGE PIER FOR TAPERED HOLLOW CIRCULAR PIER AND STRAIGHT HOLLOW CIRCULAR PIER

Height (m)	Base Bending Moment (kN.m)	Volume of Concrete Required		Cost of Construction of Pier	
		Straight Hollow Circular pier (m^3)	Tapered Hollow Circular pier (m^3)	Straight Hollow Circular pier (Rs. lakhs)	Tapered Hollow Circular pier (Rs. lakhs)
15.00	6823	67.27	42.26	7.80	5.92
20.00	12238	123.30	70.25	14.30	9.84
25.00	19027	198.79	105.89	23.06	14.82
30.00	27537	297.49	150.40	34.51	21.06
35.00	37608	419.88	203.93	48.71	28.55
40.00	48918	564.74	265.74	65.51	37.20

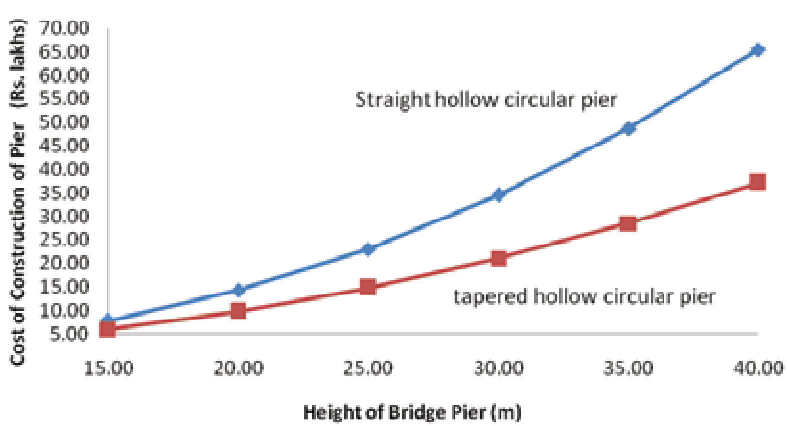

Fig. 4.5 Variation in cost comparison for different height of bridge pier for Tapered hollow circular pier and straight hollow circular pier

Cost of construction of pier obviously increases as the height of bridge pier increases. The rate of increase in cost of construction for Tapered hollow circular pier is milder in comparison with straight hollow circular pier. (Refer Table III & Figure 4.5)

V. Conclusion

1. The Volume of concrete required obviously increases with increase in Height of bridge pier.
2. The rate of increase of volume of concrete required is milder for Tapered hollow circular pier in comparison with straight Hollow circular pier.
3. The slenderness ratio is smaller for tapered hollow circular pier in comparison with straight hollow circular pier.
4. As the height of the bridge pier increases the side slope of the tapered hollow circular pier decreases.
5. As the height of bridge pier increases the cost of construction for bridge pier increases.
6. The cost of construction of tapered hollow circular pier is less in comparison with straight hollow circular pier.
7. It can be concluded that as the height of pier increases the straight hollow circular bridge pier proves to be uneconomical as compared to Tapered hollow circular bridge pier.

VI. Notations

Q_1, Q_2 axial loads in lateral direction on beam
δ_1, δ_2 deflection in lateral direction of beam
P axial load
W_T Lateral load intensity at top of pier
W_B Lateral load intensity at bottom of pier
H Height of pier
W_x Lateral load intensity at a general section 'x' on pier
M_A applied bending moment at hinged support
F Lateral thrust at top of pier
C_r correction factor
l_{eff} effective length of pier
r_{min} radius of gyration
y deflection
y_c complementary solution
y_p particular solution
FOA first order analysis
SOA second order analysis
A, B, C, D constants of integration
R_T, R_{T1}, R_{T2} reactions developed at top support of the pier

References

[1] D. Johnson Victor, "Essentials of Bridge Engineering," Oxford and IBH Publishing Company Pvt. Ltd. New Delhi.

[2] V. K. Raina "R. C. section subjected to axial load and any axis bending," *Bridge and Structure Journal,* Vol.5, pp. 126-140, 1975.

[3] V. K. Raina, "Concrete Bridge Practice- Analysis, Design and Economics," Tata McGraw-Hill Publishing Company Ltd., New Delhi, 1994.

[4] S. P. Timoshenko and J. M. Gere, "Theory of Elastic Stability," McGraw-Hill Company Ltd. Tokyo, 1961.

[5] Standard specification and code of practice for Road Bridges, Section-VII, IRC-78, Foundation and substructure, *The Indian Road Congress,* New Delhi.

[6] Standard specification and code of practice for Road Bridges, Section-III, IRC-21, Cement concrete (Plain and Reinforced), *The Indian Road Congress,* New Delhi.

[7] Standard specification and code of practice for Road Bridges, Section-II, IRC-6, Load and Stresses, *The Indian Road Congress,* New Delhi.

[8] Standard specification and code of practice for Road Bridges, Section-I, IRC-5, General features of Design, *The Indian Road Congress,* New Delhi.

[9] P. Purushothaman, "Reinforced Concrete Structural Elements-Behavior, Analysis and Design," Tata McGraw-Hill Publishing Company Ltd., New Delhi,1984.

[10] Indian Standard Code of Practice for Plain and Reinforced Concrete IS: 456-2000, Indian Standard Institution, New Delhi.