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Control Techniques for Synchronizing the States of Two Coupled Van der Pol Oscillators

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***Abstract* - Mathematical models of nonlinear oscillators are used to describe a wide variety of physical and biological phenomena that exhibit self-sustained oscillatory behavior. When these oscillators are strongly driven by forces that are periodic in time, they often exhibit a remarkable ''mode-locking'' that synchronizes the nonlinear oscillations to the driving force. Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states and is characterized by their amplitude and their phase. Their interactions can result in a systematic process of synchronization which is the adjustment of rhythms of oscillating objects due to an interaction and is quite distinct from a simple stimulus response pattern. Oscillators respond to stimuli at some times in their cycle and may not respond at others. Many important physical, chemical and biological systems are composed of coupled nonlinear oscillators. The Van der Pol equation has been used to model a number of biological processes such as the heartbeat, circadian rhythms, biochemical oscillators, and pacemaker neurons. Two such resistively coupled Van der Pol oscillators are analyzed and the phenomenon of synchronization between the states of the coupled oscillators is explored. Several control techniques to achieve synchronization are designed, implemented and performance evaluation carried out by simulation using MATLAB Software.**
***Keywords:* Nonlinear Oscillator, States of Coupled Oscillators, Van der Pol Equation, Synchronization, Control Technique**

I. INTRODUCTION

An oscillator is any system that exhibits periodic behavior. A single oscillator traces out a simple path in phase space. When two or more oscillators are coupled, however the ranges of possible behaviors become much more complex. The equations governing their behavior also tend to become intractable. Each oscillator may be coupled only to a few immediate neighbors or to all the oscillators in an enormous community. Synchrony is the most familiar mode of organization for coupled oscillators. Synchronization is the adjustment of rhythms of oscillating objects due to an interaction. Multiple periodic processes with different natural frequencies come to acquire a common frequency and in some cases also a common phase as a result of their mutual influence [1]. The Van der Pol oscillator is considered by many researchers as the nodes for various networks which are inherently unstable at the zero equilibrium [2]. Van der Pol's equation is used as a model for numerous biological oscillators [3].

The differential equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0, \quad \varepsilon>0 \qquad (1)$$

is called the Van der Pol oscillator, where 'x' is the position coordinate, which is a dynamical variable, and 'ε' is a scalar parameter which controls the nonlinearity and the strength of the damping. It is a model of a non-conservative system in which energy is added to and subtracted from the system in an autonomous fashion, resulting in a periodic motion called a limit cycle. It can be seen that the sign of the damping term, $\varepsilon(x^2 - 1)\dot{x}$ changes, depending upon whether $|x|$ is larger or smaller than unity. Numerical integration of equation (1) shows that every initial condition, (except $x = \dot{x} = 0$) approaches a unique periodic motion.

The nature of this unique periodic motion (limit cycle) is dependent on the value of 'ε'. For small values of 'ε' the motion is nearly sinusoidal, whereas for large values it is a relaxation oscillation. In a nonlinear system such as the Van der Pol oscillator, the stability is very much dependent on the input and also the initial state. Further, the nonlinear systems may exhibit limit cycles which are self-sustained oscillations of fixed frequency and amplitude. Once the system trajectories converge to a limit cycle, it will continue to remain in the closed trajectory in the state space identified as limit cycles. A limit cycle represents a steady state oscillation, to which or from which all trajectories nearby will converge or diverge. The limit cycles describe the amplitude and period of a self-sustained oscillation and they are periodic motions exhibited only by nonlinear, non-conservative systems. A limit cycle is stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle [4,5].

II. VALIDATION FOR THE NON-LINEARITY

To validate for the nonlinearity of the Van der Pol oscillator, consider equation (1).

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0 \qquad (1)$$

Let

$$x = x1 \qquad (1.1)$$

$$\dot{x} = \dot{x1} = x2 \qquad (1.2)$$

$$\ddot{x} = \ddot{x1} = \dot{x2} \qquad (1.3)$$

Substituting the above conditionswe get,

$$\dot{x2} + \varepsilon(1 - x1^2)x2 + x1 = 0 \qquad (1.4)$$

Thus,

$$\dot{x1} = x2 \quad (1.5)$$

$$\dot{x2} = \varepsilon(1 - x1^2)x2 - x1 \quad (1.6)$$

The Jacobian Matrix which is

$$A = \begin{bmatrix} \frac{\partial f1}{\partial x1} & \frac{\partial f1}{\partial x2} \\ \frac{\partial f2}{\partial x1} & \frac{\partial f2}{\partial x2} \end{bmatrix}$$

can be now obtained as,

$$A = \begin{bmatrix} 0 & 1 \\ 2\varepsilon x1x2 - 1 & \varepsilon - \varepsilon x^2 \end{bmatrix} \quad (1.7)$$

A. Singular Points: A system represented by an equation, $\dot{X} = F(X)$, is an autonomous system. For such a system, consider the points in the phase space at which the derivative of all the state variables are zeros, such point are called as singular points. If the system is placed at such a point, it will continue to lie there if left undisturbed (as the derivative of all the phase variable being zero, the system states remain unchanged) [5].

$$\text{Let} \quad \dot{x1} = \dot{x2} = 0 \quad (1.8)$$

Substituting in equations 1.5 and 1.6, we get

$$x1 = x2 = 0$$

So, the singular points for this system are (0, 0).
For finding the poles, $|\lambda I - A| = 0$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2\varepsilon x1x2 - 1 & \varepsilon - \varepsilon x^2 \end{bmatrix} \right| = 0$$

where $x1 = 0$, $x2 = 0$, $\varepsilon = 0.1$.

The equation obtained is,

$$\lambda^2 - 0.1\lambda + 1 = 0 \quad (1.9)$$

and the λ value is calculated as,

$\lambda = 0.05 \pm 0.99j$

The poles $\lambda 1$ and $\lambda 2$ are complex conjugates and they lie in the left-half of the plane, according to the nature of this response the system exhibits a stable focus.

B. Limit Cycle: A system to approach a periodic behavior, which will thus, appears a closed curve in phase plane is called a Limit Cycle. The oscillatory circuit behavior is related from a mathematical point of view to the so called limit cycles. The limit cycle describes the oscillations of nonlinear system and is called stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle [5, 6, 7]. A limit cycle in a nonlinear system describes the amplitude and period of a self-sustained oscillation. Limit cycles are oscillators of the response (or output) of nonlinear system with fixed amplitude and frequency. The limit cycle representation in phase plane has been carried out previously for varied initial conditions and the results inferred for the existence of a stable limit cycle exhibiting a sustained oscillation with constant amplitude [8].

III. COUPLED VAN DER POL OSCILLATORS

Two simulated Van der Pol oscillators (VDP) are coupled by means of resistive coupling and the response of the coupled pair for various coupling strengths has been analyzed earlier [8]. The variation in the parameter 'x' of the oscillator model which is the position coordinate, and also a dynamical variable is now carried out with $x1 = 1$ and $x2 = 0.8$ ($x1 > x2$) and the simulation result presented in Fig.1. Similarly for the position coordinates $x1 = 1$ and $x2 = 1.2$ ($x1 < x2$) the simulation result is presented in Fig.2. The coupling conductance of G_C=1/100 S (Siemens) is considered throughout the analysis as it yields a small phase shift between the states of the coupled pair of oscillators [8].

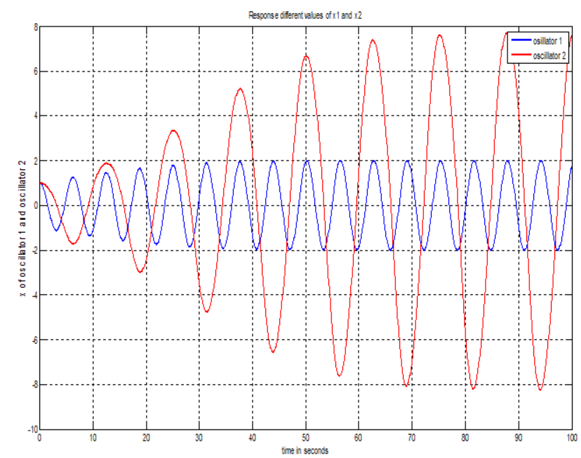

Fig.1 Response of position coordinate 'x' for $x1 > x2$

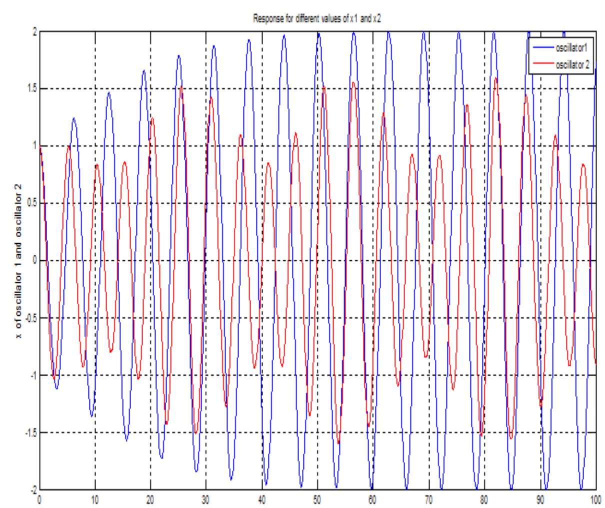

Fig.2 Response of position coordinate 'x' for $x1 < x2$

From the above two responses it is apparent that for any variations in the position coordinate 'x', the state of the coupled oscillators fail to be in synchrony with substantial amplitude and phase differences. To establish synchrony between the coupled oscillators both in amplitude and phase, several control techniques are proposed, the same designed and implemented.

IV. CONTROL TECHNIQUES

A. Proportional Control: A Proportional Control scheme is implemented with a position coordinate 'x' as 1 effected equally for both the coupled identical Van der Pol oscillators (VDP) for which, the state of the oscillators attain synchrony as shown in Fig.3. The controller parameters are estimated from the cost function plot where the value of the controller parameters that yields the lowest ISE (Integral Square Error) value is selected [8].

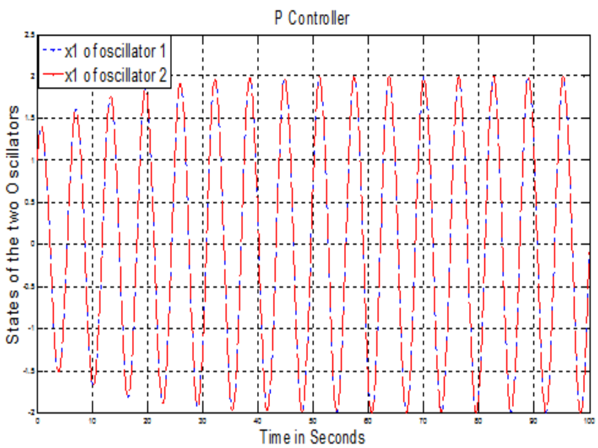

Fig.3 Coupled VDP oscillators in synchrony with P Control for $x1$=$x2$

A further decrease made in the position coordinate $x2$ of the second oscillator, while maintaining $x1$ as before ($x1$>$x2$), also synchronizes the coupled pair with P control, as shown in the Fig.4. Similarly for $x1$<$x2$ the P control when implemented eventually maintain synchrony between the states of the two coupled oscillators as shown in Fig.5.

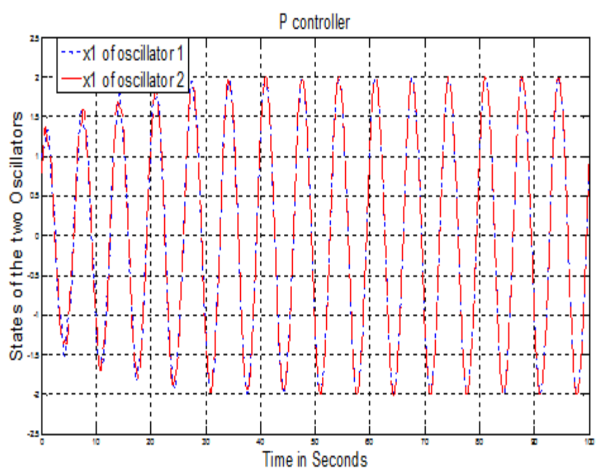

Fig.4 Coupled VDP oscillators in synchrony with P Control for $x1$>$x2$

For performance evaluation, in addition to ISE (Integral Square Error), the synchronization time (both in amplitude and phase) along with steady state error are considered and the same is tabulated for all variations effected throughout and are presented at the end in Tables I, II and III.

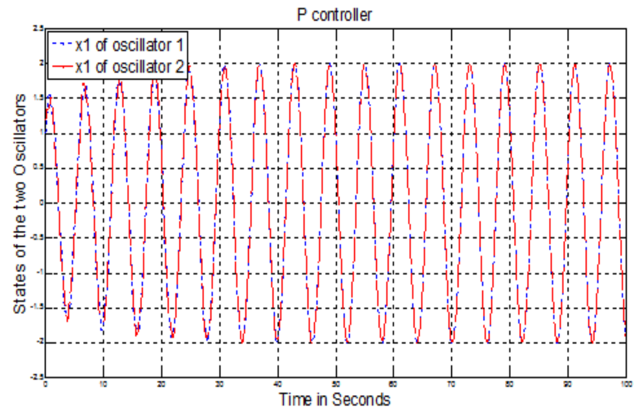

Fig.5 Coupled VDP oscillators in synchrony with P Control for $x1$<$x2$

B. Proportional+ Integral Control: With identical position coordinate 'x' maintained as before, a PI control implemented is seen to synchronize the two coupled oscillators and the simulated response is presented in Fig.6. As before the controller parameters are estimated from the cost function plot where the value of the controller parameters that yields the lowest ISE value is selected [8]. The same procedure is repeated for position coordinates $x1$>$x2$ and also for $x1$<$x2$. The simulated responses where the states of the coupled VDP oscillators are eventually in perfect synchrony in both the phase and amplitude are depicted in Fig.7 and Fig.8 respectively.

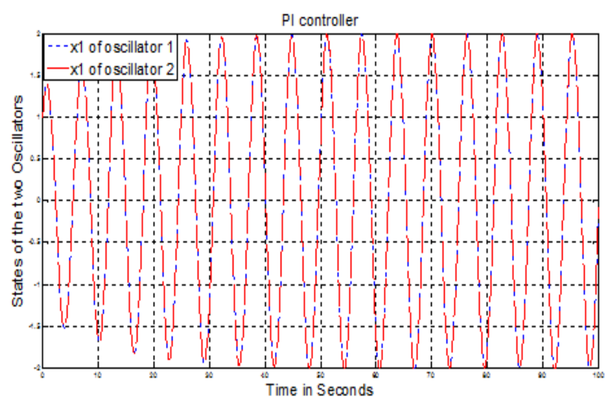

Fig.6 Coupled VDP oscillators in synchrony with PI Control for $x1$=$x2$

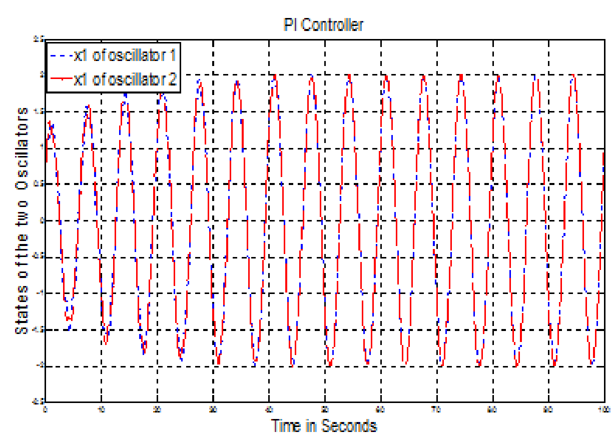

Fig.7 Coupled VDP oscillators in synchrony with PI Control for $x1$>$x2$

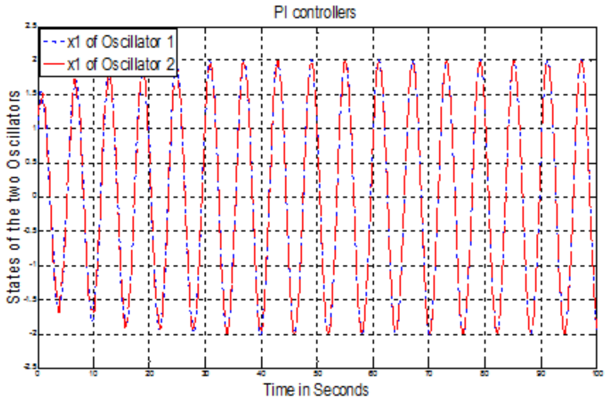

Fig.8 Coupled VDP oscillators in synchrony with PI Control for $x1<x2$

C. Proportional+Derivative Control: With the same position coordinate 'x' common to both the coupled oscillators as before, a PD control is implemented which establishes synchrony and the simulated response is depicted in Fig.9. The controller parameters are estimated from the cost function plot with the procedure repeated for position coordinates $x1>x2$ and for $x1<x2$ [8]. The simulated responses are depicted in Fig.10 and Fig.11 correspondingly. In all the responses the states of the coupled VDP oscillators are observed to be in perfect synchrony with respect to both phase and amplitude

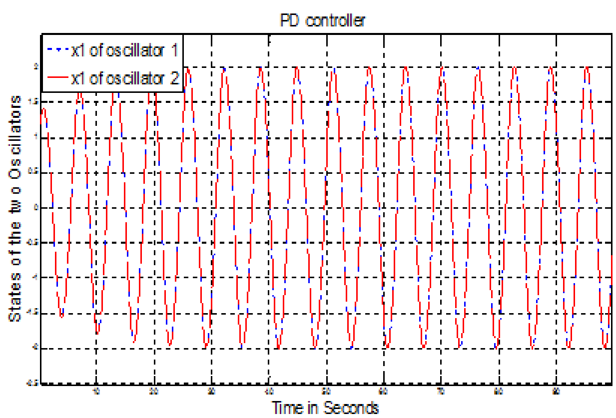

Fig.9 Coupled VDP oscillators in synchrony with PD Control for $x1=x2$

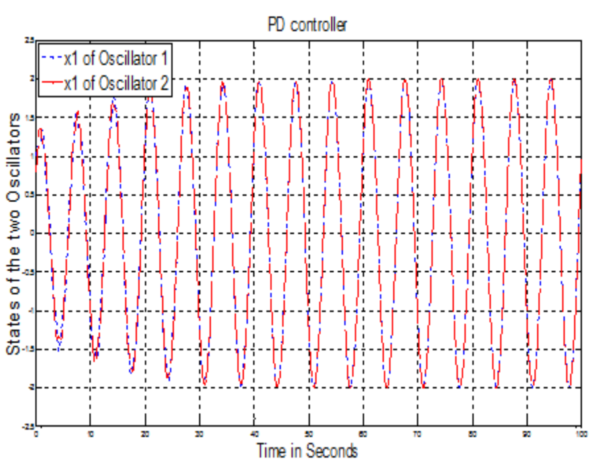

Fig.10 Coupled VDP oscillators in synchrony with PD Control for $x1>x2$

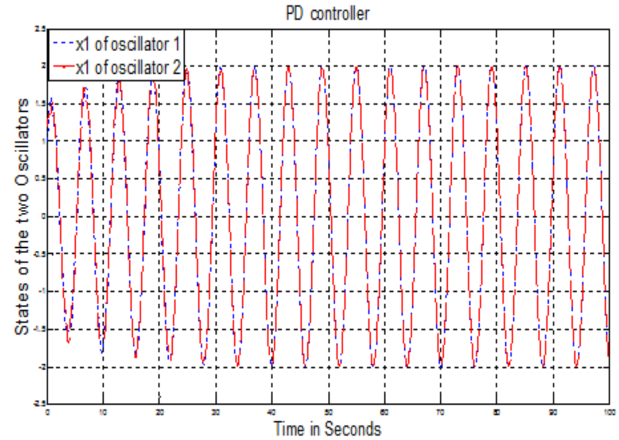

Fig.11 Coupled VDP oscillators in synchrony with PD Control for $x1<x2$

D. Proportional+Integral+Derivative Control: The cost function plot is again made use of to find the controller parameters for PID scheme [8]. The position coordinate 'x' is maintained the same for both the coupled oscillators and PID control is implemented which synchronizes the states of the two coupled oscillators. The simulated response is given in Fig.12. The procedure is repeated for two conditions where the position coordinates are $x1>x2$ and $x1<x2$. The simulated responses are provided in Fig.13 and Fig.14. In all the responses the states of the coupled oscillators are in perfect synchrony.

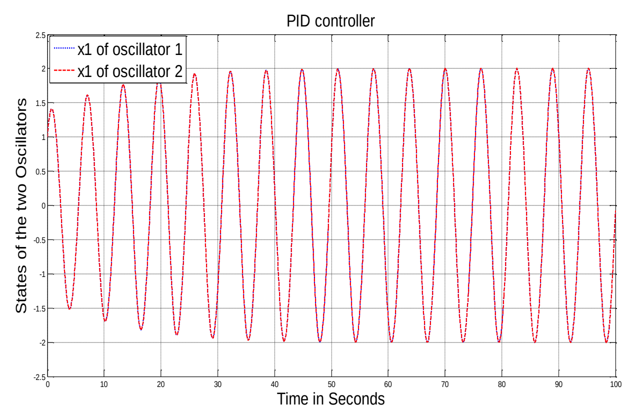

Fig.12 Coupled VDP oscillators in synchrony with PID Control for $x1=x2$

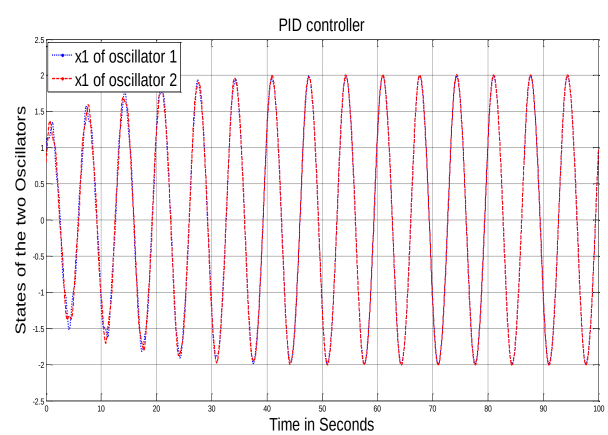

Fig.13 Coupled VDP oscillators in synchrony with PID Control for $x1>x2$

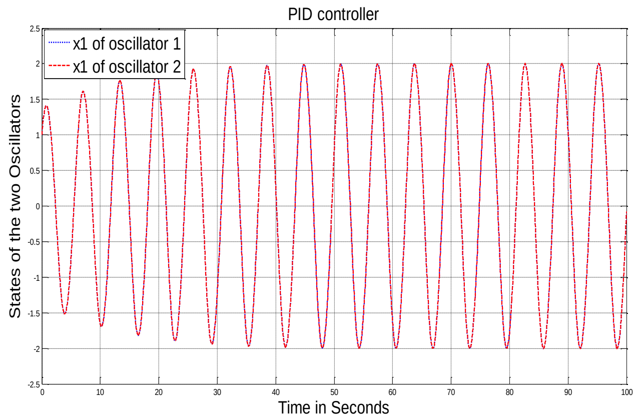

Fig.14 Coupled VDP oscillators in synchrony with PID Control for $x1<x2$

E. Fuzzy Logic Control: A Mamdani type Fuzzy Logic Control (FLC) is designed with triangular membership function with the appropriate rule base as seen in the rule viewer of Fig.15 [8].

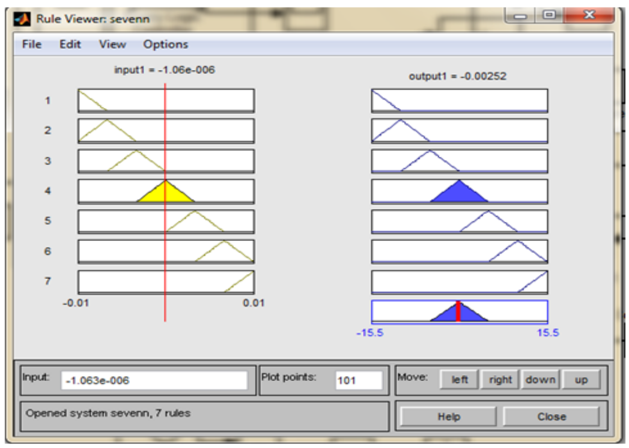

Fig.15 Rule viewer in the implemented Fuzzy Logic Control

The FLC is implemented to maintain synchrony between the states of the coupled oscillators for identical position coordinate 'x' as shown in Fig.16.

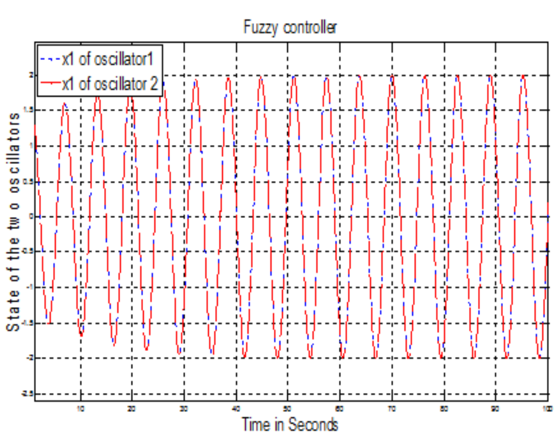

Fig.16 Coupled VDP oscillators in synchrony with Fuzzy Logic Control for $x1=x2$

The simulation is repeated for two more conditions as before where the position coordinates are maintained such that $x1>x2$ and $x1<x2$. The simulated responses are provided in Fig.17 and Fig.18. All the simulated responses are seen to possess perfect synchrony among the states of the coupled VDP oscillators.

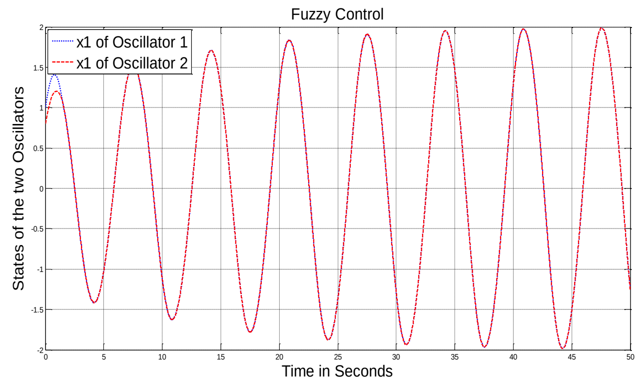

Fig.17 Coupled VDP oscillators in synchrony with Fuzzy Logic Control for $x1>x2$

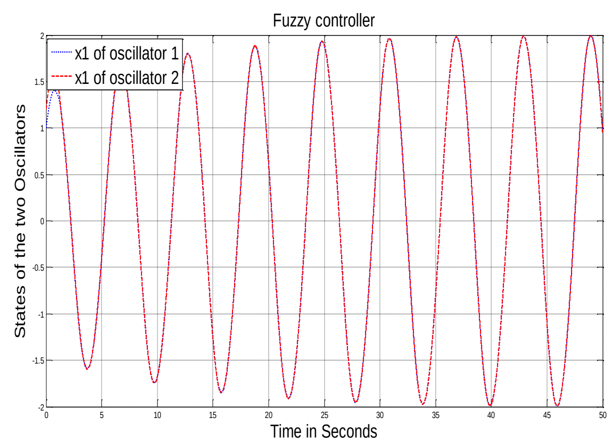

Fig.18 Coupled VDP oscillators in synchrony with Fuzzy Logic Control for $x1<x2$

V. PERFORMANCE EVALUATION

The problem of designing the 'best' controller can be formulated as the type of the controller and the values of its adjusted parameters so as to minimize the ISE (Integral of the Square Error) of the system's response [9]. To strongly suppress large errors, ISE is better because the errors are squared and thus contribute more to the value of the integral and hence the same is opted in this work where,

$$\text{ISE} = \int_0 E^2(t)\,dt$$

and E(t) is the error. In addition to ISE, steady state errors as well as the synchronization time (both in amplitude and phase) are considered and the same tabulated for all variations effected.

The notations 'X' and 'V' in Tables I, II and III are arrived from equation (1) which are considered as the first and second state derivatives respectively.

TABLE I POSITION COORDINATE $x1 = x2$

S.No.	Control Technique	$x1$	$x2$	$G_C(S)$	ISE	Synchronization time (Seconds)
1	P	1	1	1/100	0.00099	X=65 V=60
2	PI	1	1	1/100	0.001007	X=65 V=65
3	PD	1	1	1/100	0.00001926	X=65 V=50
4	PID	1	1	1/100	0.00001542	X=65 V=65
5	FUZZY	1	1	1/100	0.00000465	X=35 V=40

TABLE II POSITION COORDINATE $x1 > x2$

S.No.	Control Technique	$x1$	$x2$	$G_C(S)$	ISE	Synchronization time (Seconds)
1	P	1	0.8	1/100	0.5477	X=65 V=60
2	PI	1	0.8	1/100	0.6381	X=65 V=65
3	PD	1	0.8	1/100	0.3831	X=65 V=50
4	PID	1	0.8	1/100	0.4313	X=65 V=65
5	FUZZY	1	0.8	1/100	0.0560	X=40 V=45

TABLE III POSITION COORDINATE $x1 < x2$

S.No.	Control Technique	$x1$	$x2$	$G_C(S)$	ISE	Synchronization time (Seconds)
1	P	1	1.2	1/100	0.3895	X=45 V=50
2	PI	1	1.2	1/100	0.4454	X=50 V=50
3	PD	1	1.2	1/100	0.2684	X=40 V=45
4	PID	1	1.2	1/100	0.2963	X=40 V=45
5	FUZZY	1	1.2	1/100	0.04806	X=30 V=35

VI. RESULTS AND DISCUSSION

The Van der pol oscillator has been validated for its Nonlinearity and the same realized using simulink tool box of MATLAB software. It is inferred from the simulation studies that all the proposed control techniques does yield zero steady state error. From the performance evaluation tables presented above for all the 3 cases where the position coordinates of the states of the coupled oscillators are kept at $x1 = x2$, $x1 > x2$ and $x1 < x2$, the Fuzzy Logic Control exhibits better performance. FLC possess the lowest ISE for all the three cases, exhibits faster response, consumes lesser time to attain synchronization among the states of the coupled van der pol oscillators both in amplitude and phase well ahead of the rest.The simulation results quantify that the proposed FLC control technique can better quickly aid the coupled VDP oscillators to undergo synchronization.

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